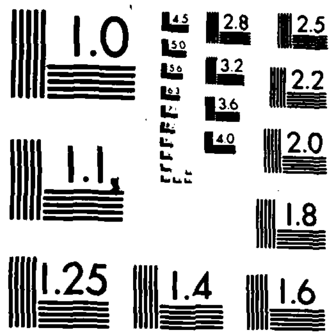
AD-A137 804 FRACTAL PHASE SCREENS(U) LA JOLLA INST CA CENTER FOR THE STUDY OF NONLINEAR DYNAMICS B J WEST JUN 83 LJI-R-83-236 AFOSR-TR-84-0053 F49620-82-C-0058 1/1

UNCLASSIFIED F/G 12/1 NL

END
FILMED
3-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

5

La Jolla
INSTITUTE

LJI-R-83-236

CENTER FOR STUDIES OF NONLINEAR DYNAMICS
8950 VILLA LA JOLLA DRIVE • SUITE 2150
LA JOLLA • CALIFORNIA 92037 • PHONE (619) 454-3581

FRACTAL PHASE SCREENS

Final Report

(Contract No. F49620-82-C-0058)

by

Bruce J. West

Center for Studies of Nonlinear Dynamics*

La Jolla Institute
8950 Villa La Jolla Drive, Suite 2150
La Jolla, CA 92037

*Affiliated with University of California, San Diego

JUNE 1983

Supported by the Air Force Office of Scientific Research

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 84-0053	2. GOVT ACCESSION NO. AD-A137 804	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Fractal Phase Screens		5. TYPE OF REPORT & PERIOD COVERED Final Report 1 May 82 - 30 April 83
		6. PERFORMING ORG. REPORT NUMBER LJI-R-83-236
7. AUTHOR(s) Bruce J. West		8. CONTRACT OR GRANT NUMBER(s) F49620-82-C-0058
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Studies of Nonlinear Dynamics La Jolla Institute 8950 Villa La Jolla Dr., Ste. 2150, La Jolla, CA 92037		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2316/A2
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Air Force Air Force Office of Scientific Research (AFOSR)/NC Bolling Air Force Base, DC 20332		12. REPORT DATE June 1983
		13. NUMBER OF PAGES 31
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

scintillation spectrum, satellite communication, scaling, fractals, diffractals, coherence, scintillation index, phase screen

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A plane wave emergent from a scattering layer is modeled as a boundary value problem. The wave is specified at $z = 0^+$ by a unit amplitude and a phase prescribed by a homogeneous isotropic random function with a power law spectrum and Gaussian statistics. It is shown that the statistics of the wave propagating in free space away from the boundary cannot be Gaussian, expect perhaps in the far field region where the scattering is saturated. It is also shown that the amplitude and intensity spectra of this wave each satisfy (over)

DD FORM 1 JAN 73 1473

UNCLASSIFIED

84 02 10 100

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. specific scaling relations. Such a wave has been called a diffractal by Berry[16] and herein we further develop some of his ideas on diffractals, in particular extending his discussion from two to three spatial dimensions. A study of the sciltillation index indicates that a diffractal is not an unreasonable model of a radio wave passing through the ionsophere, as it does in satellite communication.

LJI-R-83-236

FRACTAL PHASE SCREENS

Final Report

(Contract No. F49620-82-C-0058)

by

Bruce J. West

Center for Studies of Nonlinear Dynamics*

La Jolla Institute
8950 Villa La Jolla Drive, Suite 2150
La Jolla, CA 92037

*Affiliated with University of California, San Diego

Supported by the Air Force Office of Scientific Research

TABLE OF CONTENTS

ABSTRACT (U)

A plane wave emergent from a scattering layer is modeled as a boundary value problem. The wave is specified at $z = 0$ by a unit amplitude and a phase prescribed by a homogeneous isotropic random function with a power law spectrum and Gaussian statistics. It is shown that the statistics of the wave propagating in free space away from the boundary cannot be Gaussian, except perhaps in the far field region where the scattering is saturated. It is also shown that the amplitude and intensity spectra of this wave each satisfy specific scaling relations. Such a wave has been called a diffractal by Berry[~~10~~ 9] and herein ~~we~~ further develops some of his ideas on diffractals, in particular extending his discussion from two to three spatial dimensions. A study of the scintillation index indicates that a diffractal is not an unreasonable model of a radio wave passing through the ionosphere, as it does in satellite communication.

1. INTRODUCTION

In this paper we investigate the statistical properties of a freely propagating plane wave whose phase is specified on a transverse plane located at z = 0. The wave is propagating in the z-direction with constant frequency ω and wavelength $\lambda(=2\pi/k)$ with amplitude $\psi(\mathbf{r}, z, t) = e^{-i\omega t}u(\mathbf{r},z)$ such that

$$u(\mathbf{r},z) = e^{ik\varphi(\mathbf{r})} \text{ at } z = 0^{+} \tag{1.1}$$

where $\varphi(\mathbf{r})$ is a function of only the transverse coordinates $\mathbf{r} = (x,y)$. In a great many geophysical situations the phase function is assumed to be random so as to describe such phenomena as the twinkling of stars due to atmospheric turbulence,[1] radio wave scintillation in satellite communication due to electron density fluctuations in the ionosphere,[2] interstellar scintillation,[3] interplanetary scintillation of quasar radio sources caused by electron density irregularities in the solar wind,[4] etc.

The observed scintillation or fluctuation in the amplitude and phase of the received wave results from interference of phase points along the wavefront in the z = 0 plane as the wave propagates away from the boundary. The interference pattern is determined at least in part by the statistical properties of $\varphi(\mathbf{r})$. Booker et al.[5] were the first to use an expression of the form (1.1) to model the effect of a localized scattering region on the subsequent propagation of a scalar wave away from the scattering region. The basic assumption they employed is that the individual scatterings that a wave undergoes as it propagates through a region of fluctuating refractive index are not important; only the net change in the observables are significant. The physical observables far from the scattering region are the change in amplitude and phase in the received wave relative to a reference wave that has not been so scattered. Thus if one is interested in only the net effect of the scattering region on the received signal, the entire interval can be represented by a thin phase screen having the desired bulk

properties, i.e. the phase screen gives rise to the same overall phase shift and amplitude variation as the physical scattering interval.

Since its introduction (1.1) has been generalized to represent an extended scattering region by building up a sequence of thin phase screens.[6-8] Near the screen the initial field fluctuations affect only the phase, but as the wave propagates in free space away from the screen an interference pattern develops. The distance to this interference pattern is dependent on the density gradients in the phase screen. The larger the density gradients the greater is the angle through which the transmitted wave is scattered and the nearer to the screen is the subsequent interference pattern. It is the small scale spatial structure in the scattering medium that has large spatial gradients[9]. For a scattering interval whose fluctuations are manifestations of some turbulent motion there is a broad spectrum of scales associated with the irregularities in the medium. For example, the spectrum of irregularities of the electron density in the ionosphere has a range of from a few meters to several hundreds of kilometers, i.e., five decades or more in spatial scale. Thus radio waves with frequencies on the order of 10 MHz and lower encounter irregularities on the same scale as their wavelength and traditional propagation models, such as geometric optics, cannot describe the transmission of the wave through the phase screen, i.e. multiple scattering effects become important.[10,11] It is remarkable therefore that weak scattering theory has been so successful in explaining the observed scintillation of radio waves in satellite communication as well as other data.[12,13]

Model calculations based on phase screen descriptions make two fundamental assumptions about the fluctuations in the medium. The first is that the *statistics of the irregularities are described by a zero-centered Gaussian distribution.* The second is that the spectrum of scales of the irregularities has a power-law form. These two assumptions however do not provide a complete

characterization of the fluctuations. One must also introduce an outer scale cut-off in order to avoid a long wavelength divergence of the integrated spectrum. Most ionospheric data can then be adequately described using weak scatter theory with a power-law spectrum of irregularities with power exponents between 3.8 and 4.8 and an outer scale cutoff of approximately 10 km.[12,13]

For a thin screen the phase in (1.1) is usually given by the straightline ray path integral across the scattering interval (weak-scatter theory.)[13]

$$\varphi(\mathbf{r}) = \int_{-\Delta L}^{0} N(\mathbf{r},z)\, dz \tag{1.2}$$

where $N(\mathbf{r},z)$ is the deviation in the refractive index from its ambient value, ΔL is the thickness of the scattering interval and the plane wave is normally incident on the ribbon of scatterers. It is clear from (1.2) that the statistics of the phase fluctuations are the same as those in the index of refraction. Thus $\varphi(\mathbf{r})$ is a Gaussian process only if the irregularities in the medium are also Gaussian. If however the mechanism leading to the irregularities in the medium is turbulence, then as is well known the statistics of the fluctuations cannot be Gaussian.[14] Therefore either $\varphi(\mathbf{r})$ is not a Gaussian process, or the source of the power-law behavior of the spectrum of N-fluctuations is not turbulence, or both. In the example of the ionosphere there is apparently no experimental evidence to support the assumption that the fluctuations in plasma density are normally distributed, although *in situ* measurements indicate the spectrum is a power-law over a restricted range of scales.[12]

It is not our intention here to provide a detailed list of the possible instability mechanisms that may contribute to the power-law spectrum of density fluctuations in the scattering interval. Rather we suggest that the assumption of Gaussian phase fluctuations has been made for mathematical convenience and has not been physically justified, at least in the case of weak scattering [cf. (1.2)]. In spite of this we retain (1.1) to represent the transmission of a wave

through a region of tenuous scatterers, modeled by a power-law phase screen and assume the statistics of $\varphi(\mathbf{r})$ are Gaussian. We show that the power-law behavior of the spectrum implies that the fluctuations in the medium satisfy a scaling relation. We assume that if γ is a real parameter then the phase function $\varphi(\gamma\mathbf{r})$ can be obtained from $\varphi(\mathbf{r})$ by the scaling relation $\varphi(\gamma\mathbf{r}) = \gamma^{d/\mu}\varphi(\mathbf{r})$, where d is the Euclidean dimension (d = 2 in the case considered here) and μ characterizes the scaling behavior. The scaling relation is a geometric constraint on the phase surface.

The scaling properties of waves transmitted through fluctuating media with Gaussian statistics and power-law spectra for the fluctuations have not been fully explored. Rino,[25] in his analysis of strong scattering in a power-law phase screen, investigated the results for the second-order statistics of intensity using a model developed by Gochelashvily and Shishov.[27] In the present report we also obtain results for second-order statistics of intensity, but we emphasize the scaling behavior of the propagated wave. The relation between the present results and the earlier work of Rino and others is commented on in the appropriate places in the text.

Mandelbrot[15] has recently drawn attention to the physical importance of functions having scaling properties in his important work on *fractals*. We apply some of his observations to the phase surface given that it is assumed to obey a scaling law. One consequence of the scaling (self-similarity) property of the phase function $\varphi(\mathbf{r})$ is that irregularities exist on all scales so that such functions do not possess derivatives, i.e. one cannot construct a tangent to the surface $\varphi(\mathbf{r})$ at any point in space. Mandelbrot not only advances the thesis that *most* physical processes are discontinuous and described by such non-differentiable functions, but in addition those processes studied by Norbert Weiner, i.e. Gaussian random functions, although of this kind are in fact benign

by comparison. Thus although Weiner processes are presently encountered in practically all physical models of phenomena in which fluctuations are thought to be important, Mandelbrot contends that the random component of nature is much richer. The experimental data he has amassed and juxtaposed to support this contention is impressive.[15]

Berry[16] has applied and extended some of the ideas of Mandelbrot in the wave propagation context by introducing the notion of a diffractal. A diffrac is a wave that has encountered a fractal object. Such an object has a self-similarity property such that it possesses structure on all scales. An incide wave of wavelength λ is fairly insensitive to structure on scales much smaller than λ and for structures much larger than λ the techniques of geometric optics can be employed to describe the wave-object interactions. However this separation of effects *cannot* be made when the scattering object is a fractal *since then the self-similar character insures* that there will be structure on a range of scales that includes λ and thus no geometric optics limit exists. Berry models a diffractal using (1.1) by assuming $\varphi(x)$, i.e. $\varphi(\mathbf{r})$ in one transverse dimension, to be a Gaussian random fractal function with a power-law spectrum having a fractal dimension D lying between 1 and 2. Physically this boundary condition constitutes a thin phase screen approximation for a wave reflected by a fractal surface or refracted by a slab of transparent material with fractal refractive index.

In Section 2 we discuss a homogeneous, isotropic statistical field as a model for the fractal phase surface. Some of the physical consequences of having a power-law spectrum, which one does for a fractal function, are discussed for this case including the relation between the power-law index and the fractal dimensionality of the statistical process. In Section 3 the paraxial approximation is used to propagate the fractal phase surface away from the edge of the phase

screen. The statistical properties of the free field are examined by calculating the coherence of the diffractal wave as well as the correlation and spectrum of the diffractal intensity. In Section 4 the scintillation index is calculated as a function of distance from the phase screen and it is determined that a diffractal cannot have Gaussian statistics except *perhaps* in the far field region where the fluctuations are saturated.

2. HOMOGENEOUS PHASE SCREEN

Herein we assume that the phase surface $\varphi(\mathbf{r})$ is a locally homogeneous random field, that is to say a random field with statistically homogeneous increments and follow the discussion of Monin and Yaglom[14] to establish its general properties. Spatial homogeneity, like time stationarity, means that the distribution of the increments $[\varphi(\mathbf{r}_1) - \varphi(\mathbf{r}_2)]$ depends only on the magnitude of separation between the two phase points, i.e. on $|\mathbf{r}_1 - \mathbf{r}_2|$. Consequently the probability density is invariant to any rigid translation of the field. It is easily demonstrated that the average of the incremental phase $[\varphi(\mathbf{r} + \mathbf{R}) - \varphi(\mathbf{r})]$ in such a case is a linear functional of $\mathbf{R}$:

$$< [\varphi(\mathbf{r} + \mathbf{R}) - \varphi(\mathbf{r})] > \; = \; \mathbf{c} \cdot \mathbf{R} \quad . \tag{2.1}$$

If we further assume that the field is locally isotropic, meaning that the probability density for the incremental phase is invariant under rotations and reflections of the spatial *increments*, then $\mathbf{c}$ is the zero vector.[14] The mean incremental phase is therefore zero.

Inasmuch as we are also assuming that the incremental phase is a Gaussian process it is completely determined by its first two moments. The general second moment of the incremental phase is

$$< [\varphi(\mathbf{r}_1 + \mathbf{R}_1) - \varphi(\mathbf{r}_1)] [\varphi(\mathbf{r}_2 + \mathbf{R}_2) - \varphi(\mathbf{r}_2)] > \; = \; D(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{R}_1, \mathbf{R}_2) \quad . \tag{2.2}$$

For a locally homogeneous, isotropic phase surface (2.2) can be written as[14]

$$D(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{R}_1, \mathbf{R}_2) \; = \; \frac{1}{2}\Big\{ D\big(|\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{R}_1|\big) + D\big(|\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{R}_2|\big) \tag{2.3}$$

$$- D\big(|\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{R}_1 + \mathbf{R}_2|\big) - D\big(|\mathbf{r}_1 - \mathbf{r}_2|\big)\Big\}$$

where $D(R)$ is the phase structure function

$$D(R) \; = \; < [\varphi(\mathbf{r} + \mathbf{R}) - \varphi(\mathbf{r})]^2 > \quad . \tag{2.4}$$

The phase structure function is as important for the incremental process as the correlation function, $\rho(\mathbf{R})$ [$\equiv \langle\varphi(\mathbf{r}+\mathbf{R})\ \varphi\ (\mathbf{r})\rangle$], is for the phase surface. From the definition (2.4) it is clear that

$$D(R) \;=\; 2\,[\,\rho(0) - \rho(R)\,] \tag{2.5}$$

so that if we know the correlation function for the process we know the structure function as well. The converse of this statement is not always true however, since a knowledge of D(R) will not uniquely determine ρ (R). In the situation we wish to consider here the correlation of the phase surface $\varphi(\mathbf{r})$ with $\varphi(\mathbf{r}+\mathbf{R})$ vanishes as $\mathrm{R} \to \infty$ so that

$$\lim_{R\to\infty} D(R) \;=\; D(\infty) \;=\; 2\rho\,(0);\;\; \rho(R) \;=\; \frac{1}{2}\Big[D(\infty) - D(R)\Big]\;. \tag{2.6}$$

Thus, for locally homogeneous isotropic random processes with a correlation function $\rho(R)$ which vanishes at infinity, the statistical characteristics of $\rho(R)$ and D(R) can each be determined from the other.

Let us now consider the spectral decomposition of the phase surface $\varphi(\mathbf{r})$ with homogeneous isotropic increments and the corresponding structure function D(R):

$$\varphi(\mathbf{r}) \;=\; \int (e^{i\,\mathbf{k}\cdot\mathbf{r}} - 1)\; d\Phi\;(\mathbf{k}) + \varphi_0\;. \tag{2.7}$$

The integral is over the entire wavevector space since $\mathrm{d}\,\Phi(\mathbf{k}) \;=\; \Phi(d\mathbf{k})$ is a random function of the wavevector $\mathbf{k}$ defined on all intervals ($\mathbf{k}$, $\mathbf{k}+d\mathbf{k}$) not containing the origin and $\varphi_0 \;=\; \varphi(0)$ is a constant random variable. The two dimensional spectral density $\Psi\;(\mathbf{k})$ of the phase surface $\varphi(\mathbf{r})$ is defined by

$$<\, d\Phi(\mathbf{k})\; d\;\Phi^*(\mathbf{k})\,> \;=\; \Psi\;(\mathbf{k})\; d^2k \tag{2.8}$$

where $\Psi(\mathbf{k})$ is a non-negative even function of the wavevector $\mathbf{k}$. The phase structure function (2.4) can be expressed in terms of the spectral density using (2.7) and (2.8) as

$$D(R) = 2\int [1 - \cos \mathbf{k}\cdot\mathbf{R}]\, \Psi(\mathbf{k})\, d^2k \quad . \tag{2.9}$$

We note that the function $\Psi(\mathbf{k})$ can rapidly diverge as $|\mathbf{k}| \to 0$ according to (2.9). However, the factor $(1 - \cos \mathbf{k}\cdot\mathbf{R})$ tends to zero as $(\mathbf{k}\cdot\mathbf{R})^2$ so that as long as

$$\int_{|\mathbf{k}|<k_0} k^2\, \Psi(\mathbf{k})\, d^2k < \infty \tag{2.10}$$

the phase structure function remains finite for finite R.

Herein we assume that the phase structure function is given by

$$D(R) = C_\mu R^\mu \; ; 0 \le \mu \le 2 \tag{2.11}$$

where C_μ is a positive constant determining the strength of the correlations and μ is a parameter to be determined. The inverse Fourier transform of (2.9), with the phase structure function (2.11), yields

$$\Psi(\mathbf{K}) = -\frac{C_\mu}{2(2\pi)^2}\int d^2R\; R^\mu e^{-i\mathbf{K}\cdot\mathbf{R}}$$

$$= \frac{2^\mu \mu}{(2\pi)^2} C_\mu \Gamma(\mu/2)\, \Gamma(1+\mu/2)\, \sin(\mu\pi/2) \,/\, |\mathbf{K}|^{\mu+2} \quad . \tag{2.12}$$

Thus we have a power-law spectrum for the incremental phase and integrating (2.12) over an unbounded region, i.e.

$$\langle\varphi^2\rangle = \int \Psi(\mathbf{K}) d^2K \tag{2.13}$$

we observe that the mean square phase surface, $\langle \varphi^2 \rangle$, is infinite. The phase surface therefore satisfies the conditions for being a Gaussian fractal function with homogeneous isotropic increments and spectral density $\Psi(\mathbf{K})$.

A fractal function $F(\mathbf{x})$ is a continuous but non-differentiable function that satisfies a scaling relation of the form $F(\gamma\mathbf{x}) = \gamma^{\mu/d} F(\mathbf{x})$ where γ is a real parameter, d is the Euclidean dimensionality of the space and μ is a real parameter. Thus in the case d = 1, if the variations in the function are known in

an interval $\gamma x_0 \geq x \geq x_0$, they can be determined in the expanded interval $\gamma^2 x_0 \geq x \geq \gamma x_0$, as well as in the contracted interval $x_0 \geq x \geq x_0/\gamma$. This scaling behavior is known as self-similarity and is one of the defining properties of fractal processes. A process described by a function the graph of which has a Hausdorff-Besicovich dimensionality greater than unity is said to be fractal, i.e. a fractal process is a process described by a function that exhibits a fractal dimensionality.[15] For example, the fractal dimensionality D of a Wiener process is two in a Euclidean space of one, two and three dimensions. In general, fractal functions (processes) with $D < 2$ and $D < d$ are not space filling. In order to maintain the self-similarity property a fractal process can only occupy space in clustered or localized patches.[17-19] The phase function $\varphi(\mathbf{r})$ is such a random fractal function in the Euclidean space $d = 2$. Its fractal (Hausdorff-Besicovich) dimensionality is shown below to be related to the parameter μ by $\mu/2 = 2\text{-}D$.

The strength of the phase structure function, i.e. the constant C_μ in (2.12), is here chosen by analogy with Berry.[16] We note that the vector connecting the end points of the phase increment $[\varphi(\mathbf{r} + \mathbf{R}) - \varphi(\mathbf{r})]$ has a finite slope, even though the function $\varphi(\mathbf{r})$ is non-differentiable. We use this fact to define the strength of the fractal as the distance over which the phase structure function has a slope of one radian, i.e.

$$D(L)/L^2 \equiv 1 \; . \tag{2.14}$$

The parameter C_μ in (2.11) can now be replaced with $L^{2-\mu}$ yielding for an arbitrary separation vector **R**,

$$D(R) = L^{2-\mu} R^{\mu} \; . \tag{2.15}$$

The notion of a fractal (Hausdorff-Besicovich) dimensionality is a useful concept for understanding the degree of irregularity possessed by a random process. We use a simple covering argument to show that the fractal dimensionality

D of the phase surface is related to the exponent μ in (2.15) and (2.12) by $D = (4 - \mu)/2$. The argument is similar in spirit to one presented by Mandelbrot,[15] wherein he uses the Lipschitz condition to determine the fractal dimensionality of a given process. Instead of the Lipschitz condition we use the result for the phase structure function (2.15).[20] We note that the number of spheres of radius R, in $(\Phi,\mathbf{r})$ -space, required to fill the volume between the phase surfaces at $\mathbf{r}$ and $\mathbf{r} + \mathbf{R}$ is

$$n = [D(R)]^{½} R^2 / R^3 \sim R^{\mu/2-1} \text{ as } R \to 0 \ . \tag{2.16}$$

The total number of spheres required to fill the volume between r = 0 and r = 1 is N = n/R as $R \to 0$. The fractal dimension D follows from the definition

$$N \sim R^D \tag{2.17a}$$

in the limit $R \to 0$ and is

$$\begin{aligned} D &= \log N / \log(1/R) \\ &= (4-\mu)/2 \end{aligned} \tag{2.17b}$$

so that D falls in the interval $1 \le D \le 2$, since from (2.11), $0 \le \mu \le 2$. Berry obtained the same result in the one dimensional case using a capacity argument. Thus the fractal dimension is independent of the Euclidean dimensionality of the $(\Phi,\mathbf{r})$ - space.

As pointed out above, the defining property of a fractal function is its self-similar character. This is also true of a random fractal function, but in this latter case the self-similarity is also manifest through the scaling of the corresponding probability density. Let $P(\Phi,\mathbf{r})$ be the probability density that the random function $\varphi(\mathbf{r})$ has a value in the interval $(\Phi, \Phi + d\Phi)$ at the location $\mathbf{r}$. Then since $P(\Phi,\mathbf{r})$ is Gaussian and $\varphi(\mathbf{r})$ has the phase structure function (2.15), we observe for a positive real parameter γ the scaling relation[20,21]

$$P(\Phi,\mathbf{r}) = P(\gamma^{\mu/2}\Phi, \gamma\mathbf{r})\,\gamma^{\mu/2} \tag{2.18}$$

where μ is the parameter in (2.15). Equation (2.18) indicates that if the phase surface $\varphi(\mathbf{r})$ is scaled in both the x and y directions by γ and in the Φ direction by $\gamma^{\mu/2}$, then the resulting phase surface is statistically indistinguishable from the original. This scaling property implies that in addition to not having a smallest scale, which is required for a fractal function, the phase function $\varphi(\mathbf{r})$ also has no largest scale and is self-correlated over an arbitrarily long distance [cf. (2.11)].[14,16]

In the case $\mu = 1$ the probability density (2.18) scales in the same way as Brownian motion; the phase structure function is proportional to R [cf. (2.11)] so that we refer to D = 1.5 as the *Brownian fractal* [cf. (2.17b)]. The case $\mu = 0$ is the *extreme fractal* where the phase surface φ is on the verge of filling a finite volume. Finally the case $\mu = 2$ is the *marginal fractal*, where the phase is "almost" a smooth function. If instead of specifying the phase structure function (2.1) we had instead originally assumed a spectrum of the form (2.12), then we could not inverse Fourier transform this spectrum to obtain (2.12) for $\mu = 2$ because the integral would diverge. Therefore we would be unable to choose the coefficient C_μ as done in (2.15). If however, we can identify C_2 as the root mean square "slope" of displacements of any length, this would replace the length L. These latter comments follow closely those of Berry.[16]

3. THREE DIMENSIONAL DIFFRACTALS

The propagation of a diffractal in free space can be determined by solving the three dimensional wave equation subject to the boundary condition $u(\mathbf{r},z) = \exp[ik\varphi(\mathbf{r})]$ at $z = 0^+$. For a plane wave of frequency ω and wavenumber k the wave equation reduces to the Helmholtz equation

$$\left\{\frac{\partial^2}{\partial z^2} + \nabla^2 + k^2\right\} u(\mathbf{r},z) = 0 \tag{3.1}$$

with the boundary condition

$$u(\mathbf{r}, z=0^+) = e^{ik\varphi(\mathbf{r})} , \tag{3.2}$$

and ∇^2 is the two dimensional Laplacian. The Helmholtz equation is an elliptic partial differential equation so that in order to find the wave field at a given point in space one must solve the equation for the field at all points in space. One often approximates the Helmholtz equation by a parabolic equation having the form of the Schrödinger equation in two space dimensions with the time replaced by the propagation coordinate:[1]

$$\left\{i\frac{\partial}{\partial z} + \nabla^2 + k^2\right\} u(\mathbf{r},z) = 0 . \tag{3.3}$$

The boundary condition (3.2) is the "initial value" of the solution to (3.3). In the parabolic approximation the normal to the phase front of the wave is assumed to remain close to the direction of incidence, i.e. the z-axis. For this reason the approximation is also referred to as the paraxial approximation. The condition for the validity of (3.3) is therefore small angle scattering and can often be expressed in terms of a correlation length. The validity of (3.3) will be discussed subsequently since its applicability to describing the propagation of diffractals is not obvious.

3.1 Average Diffractal

The solution to the parabolic equation (3.3) subject to the boundary condition (3.2) is given by the elementary diffraction integral

$$u(\mathbf{r},z) = e^{ikz}\,\frac{k}{2\pi z}\int d^2r'\; e^{i\frac{k}{2z}|\mathbf{r}-\mathbf{r}'|^2}\, e^{ik\varphi(\mathbf{r}')} \quad . \tag{3.4}$$

Equation (3.4) describes the diffraction of the phase front away from the phase screen point $(\mathbf{r}',0)$ to the observation point $(\mathbf{r},z)$. The average diffractal detected at $(\mathbf{r},z)$ is determined by an average over an ensemble of realizations of the phase irregularities:

$$<u(\mathbf{r},z)> = \frac{k}{2\pi z}e^{ikz}\int d^2r'\; e^{i\frac{k}{2z}|\mathbf{r}-\mathbf{r}'|^2} < e^{ik\varphi(\mathbf{r}')}> \quad . \tag{3.5}$$

However, the Gaussian nature of the phase fluctuations allows us to write

$$< e^{ik\varphi(\mathbf{r}')}> = e^{-k^2<\varphi^2>/2} = 0 \tag{3.6}$$

where the last equality follows from the fractal nature of $\varphi(\mathbf{r})$, i.e., the mean square phase variation of a fractal is infinite [cf. (2.12) and (2.13)].

3.2 Coherence of diffractals

The coherence between the diffractal wavefield (3.4) at the points $(\mathbf{r}+\mathbf{R},z)$ and $(\mathbf{r},z)$ is given by the coherence function

$$< \Gamma(\mathbf{R},z)> \equiv < u(\mathbf{r}+\mathbf{R},z)\, u^*(\mathbf{r},z) >$$

$$= \left(\frac{k}{2\pi z}\right)^2 \int d^2r' \int d^2r''\; e^{i\frac{k}{2z}[|\mathbf{r}+\mathbf{R}-\mathbf{r}'|^2-|\mathbf{r}-\mathbf{r}''|^2]}$$

$$\cdot < e^{ik[\varphi(\mathbf{r}')-\varphi(\mathbf{r}'')]}> \quad . \tag{3.7}$$

Using the conditions of Gaussianity, isotropy and homogeneity we evaluate the average in (3.7) using (2.4) to obtain after an elementary transformation of variables

$$<I(\mathbf{R}z)> = \left[\frac{k}{2\pi z}\right]^2 \int d^2\eta \int d^2\xi \, e^{-k^2 D(\xi)/2} \tag{3.8}$$

$$\cdot \exp\left\{ i\frac{k}{z}(\mathbf{r}-\boldsymbol{\eta})\cdot(\mathbf{R}-\boldsymbol{\xi}) + i\frac{k}{2z}\mathbf{R}\cdot(\mathbf{R}-\boldsymbol{\xi})\right\} \cdot$$

The integral over η yields $(2\pi)^2\,\delta\,(\mathbf{R}-\xi)\,z^2/k^2$ so that using (2.15), (3.8) reduces to

$$<I(\mathbf{R})> = \exp\left\{ -k^2 L^{2-\mu}|\mathbf{R}|^{\mu}/2\right\} \tag{3.9}$$

which is independent of z. The coherence function (3.9) is consistent with the result first obtained by Booker et. al.[5] for a thin phase screen with Gaussian statistics. Most recently Berry has obtained the one dimensional form of (3.9) for a diffractal.[16]

The energy at each scale in the diffractal is determined by the power spectrum $P_u(\mathbf{K})$ and is obtained by taking the Fourier transform of the coherence function (3.9):

$$P_u(\mathbf{K}) = \int d^2u \, e^{i\mathbf{K}\cdot\mathbf{u}} \exp\left\{ -k^2 L^{2-\mu}|\mathbf{u}|^{\mu}/2\right\} . \tag{3.10}$$

Note that (3.10) has the form of a radially symmetric Lévy stable probability density in two dimensions, that is to say that (3.9) has the form of a Lévy characteristic function with parameter μ.[19-21] A Lévy stable process is one whose distribution is a member of the class of infinitely divisible distributions and therefore it satisfies a number of scaling relations. The identification of the power spectrum of the diffractal with such a process is not accidental; rather it emphasizes the lack of a fundamental scale in the diffractal spectrum. The scaling properties of the spectrum are made evident by scaling the variables in the integrand of (3.10) by $[k^2 L^{2-\mu}/2]^{1/\mu}$ and integrating over angle to obtain

$$P_u(\mathbf{K}) = 2\pi\left[\frac{2}{k^2 L^{2-\mu}}\right]^{\frac{2}{\mu}} \int_0^{\infty} y\,dy\,e^{-y^{\mu}} J_0\left[\left(\frac{2}{k^2 L^{2-\mu}}\right)^{1/\mu}|\mathbf{K}|y\right] \tag{3.11}$$

where $J_0(\cdot)$ is the zero under Bessel function.

In the limit $K \to 0$ the Bessel function in the integrand of (3.11) is replaced by unity and we obtain, in terms of the fractal dimensionality (2.17b),

$$\lim_{K \to 0} k^2 P_u(\mathbf{K}/k) = \left[\frac{2}{k^2 L^2}\right]^{\frac{D-1}{2-D}} 2\pi\, \Gamma\,[(3-D)/(2-D)] \quad (3.12)$$

Note that (3.12) is independent of spatial wavenumber and that the spectral level for large scale features scales as $\lambda^{2/(2-D)}$. The fractal dimension D = 2 is somewhat special and will be discussed in Section 4. In the short scale asymptotic limit, $K \to \infty$, the well known asymptotic expansion for a stable Lévy process in two dimensions applied to (3.11) yields[20,21]

$$\lim_{K \to \infty} k^2 P_u(\mathbf{K}/k) = 2^{5-2D} (kL)^{2(D-1)} \Gamma^2(3-D) \sin(2-D)\pi / |\mathbf{K}/k|^{2(3-D)} \quad (3.13)$$

For intermediate values of the wave number a more detailed analysis of the integral (3.11) is required.[22,23]

The asymptotic results for the diffractal spectrum (3.12) and (3.13) can be used as an *a posteriori* justification for the paraxial approximation. The parabolic equation is an adequate description of the propagation away from the phase screen where the scattering angle, as measured by $|\mathbf{K}|/k$, is small. Therefore we require that most of the energy in the scattered wave field is in the forward direction. More quantitatively, if $|\mathbf{K}/k|$ is large, we require that $P_u(0) \gg P_u(\mathbf{K}/k)$:

$$\frac{P_u(\mathbf{K}/k)}{P_u(0)} \sim (kL)^{\frac{(4-\mu^2)}{\mu}} / |\mathbf{K}/k|^{\mu+2} \ll 1 . \quad (3.14)$$

Therefore we introduce the parameter Θ given by

$$\Theta = (kL)^{\frac{2-\mu}{\mu}} = (kL)^{\frac{D-1}{2-D}} \quad (3.15)$$

and the paraxial approximation is a good one if $k\,\Theta/|\mathbf{K}| \ll 1$, or more strongly if $\Theta \ll 1$. Thus for a given incident wavelength λ for L sufficiently small the

dominant diffracted waves are paraxial. The scattering parameter Θ increases with increasing wavenumber (decreasing wavelength) of the incident wave indicating the small scale divergence of the slope of the phase surface.

3.3 Coherence of diffractal intensity

The coherence between the diffractal intensities $|u(\mathbf{r}+\mathbf{R},z)|^2$ and $|u(\mathbf{r},z)|^2$ is given by

$$< I_2(\mathbf{R},z)> \;=\; \left(\frac{k}{2\pi z}\right)^4 \int \prod_{j=1}^{4} d^2\mathbf{r}_j$$

$$\cdot \exp\left\{ i\,\frac{k}{2z}\left(|\mathbf{r}+\mathbf{R}-\mathbf{r}_1|^2 - |\mathbf{r}+\mathbf{R}-\mathbf{r}_2|^2 + |\mathbf{r}-\mathbf{r}_3|^2 - |\mathbf{r}-\mathbf{r}_4|^2\right)\right\}$$

$$\cdot < \exp i\,k\left[\varphi(\mathbf{r}_1) - \varphi(\mathbf{r}_2) + \varphi(\mathbf{r}_3) - \varphi(\mathbf{r}_4)\right] > \qquad (3.16)$$

The average in (3.16) requires the specification of the mean square four point difference in the phase surface

$$< \left[\varphi(\mathbf{r}_1) - \varphi(\mathbf{r}_2) + \varphi(\mathbf{r}_3) - \varphi(\mathbf{r}_4)\right]^2 > \;=\; D(|\mathbf{r}_1-\mathbf{r}_2|) + D(|\mathbf{r}_1-\mathbf{r}_4|) + D(|\mathbf{r}_3-\mathbf{r}_4|)$$

$$+ D(|\mathbf{r}_3-\mathbf{r}_2|) - D(|\mathbf{r}_1-\mathbf{r}_3|) - D(|\mathbf{r}_2-\mathbf{r}_4|) \qquad (3.17)$$

so that using the local homogeneity and isotropy assumption on the phase structure function [cf. (2.15)] we obtain

$$< I_2(\mathbf{R},z)> \;=\; \left(\frac{k}{2\pi z}\right)^4 \int \prod_{j=1}^{4} d^2 r_j$$

$$\cdot \exp\left\{ i\,\frac{k}{2z}\left[|\mathbf{r}+\mathbf{R}-\mathbf{r}_1|^2 - |\mathbf{r}+\mathbf{R}-\mathbf{r}_2|^2 + |\mathbf{r}-\mathbf{r}_3|^2 - |\mathbf{r}-\mathbf{r}_4|^2\right]\right\}$$

$$\cdot \exp\left\{ -\frac{1}{2}k^2 L^{2-\mu}\left[|\mathbf{r}_1-\mathbf{r}_2|^\mu + |\mathbf{r}_1-\mathbf{r}_4|^\mu + |\mathbf{r}_3-\mathbf{r}_4|^\mu\right.\right.$$

$$\left.\left. + |\mathbf{r}_3-\mathbf{r}_2|^\mu - |\mathbf{r}_1-\mathbf{r}_3|^\mu - |\mathbf{r}_2-\mathbf{r}_4|^\mu\right]\right\} \qquad (3.18)$$

The linear transformation of variables:

$$\eta_1 = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4), \quad \eta_2 = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3 + \mathbf{r}_4),$$

$$\eta_3 = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4) \text{ and } \eta_4 = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4)$$

enables us to directly integrate (3.18) over η_1 to obtain $(2\pi)^2\,\delta(\eta_4)\,z^2/k^2$ so that integrating over η_4 (3.18) reduces to

$$<I_2(\mathbf{R},z)> = \left[\frac{k}{2\pi z}\right]^2 \int d^2\eta_2 \int d^2\eta_3\; e^{-i\frac{k}{z}\eta_2\cdot(\mathbf{R}-\eta_3)}$$

$$\cdot \exp\left\{-\frac{1}{2}k^2L^{2-\mu}\left[2\eta_2^{\mu} + 2\eta_3^{\mu} - |\eta_2+\eta_3|^{\mu} - |\eta_2-\eta_3|^{\mu}\right]\right\}. \qquad (3.19)$$

A more compact form of (3.19) is obtained by introducing the dimensionless parameter

$$\zeta \equiv kz\,\Theta^2 \qquad (3.20)$$

and using the dimensionless variables

$$\mathbf{u} \equiv k\,\Theta\,\eta_2/\zeta^{1/2};\quad \mathbf{v} = k\,\Theta\,\eta_3/\zeta^{1/2} \qquad (3.21)$$

we obtain

$$< I_2(\mathbf{R},z) > = \frac{1}{(2\pi)^2}\int d^2u \int d^2v$$

$$\cdot \exp\left\{i\mathbf{u}\cdot\left[\mathbf{v} - \mathbf{R}k\,\Theta/\zeta^{1/2}\right] - \zeta^{\mu/2}f(\mathbf{u},\mathbf{v})\right\} \qquad (3.22)$$

where

$$f(\mathbf{u},\mathbf{v}) = |\mathbf{u}|^{\mu} + |\mathbf{v}|^{\mu} - \tfrac{1}{2}|\mathbf{u}+\mathbf{v}|^{\mu} - \tfrac{1}{2}|\mathbf{u}-\mathbf{v}|^{\mu}. \qquad (3.23)$$

The spectral density of the diffractal intensity in terms of the Fourier transform of (3.22) can be written

$$P_I(\mathbf{K},\zeta) = \int d^2R\; e^{i\mathbf{K}\cdot\mathbf{R}}\left\{<\left[|u(\mathbf{r}+\mathbf{R},z)|^2 - <|u(\mathbf{r}+\mathbf{R},z)|^2>\right]\right.$$

$$\left.\cdot\left[|u(\mathbf{r},z)|^2 - <|u(\mathbf{r},z)|^2>\right]>\right\}$$

$$= \int d^2R \, e^{i\mathbf{K}\cdot\mathbf{R}} \left\{ < I_2(\mathbf{R},\zeta) > - 1 \right\} \qquad (3.24)$$

where we have used (3.9), i.e. $<|u|^2> = 1$. In terms of the scaled wavevector $\mathbf{q} = \mathbf{K}/k\Theta$ we can, after integrating over $\mathbf{R}$, rewrite (3.24) as

$$k^2 P_I(\mathbf{q}k\,\Theta,\zeta) = \frac{1}{\Theta^2} \int d^2v \cos(\mathbf{v}\cdot\mathbf{q}) \left[e^{-f(\mathbf{v},\zeta\mathbf{q})} - 1 \right] . \qquad (3.25)$$

The spectrum of intensity fluctuations therefore has the following limiting forms obtained by Gochelashvily and Shishov.[22]

In the low frequency limit $|\mathbf{K}|/k\,\Theta < \zeta^{-1}$, the exponential in (3.25) can be expanded and the first order correction to the spectrum given by

$$\lim_{q\to 0} k^2 P_I(\mathbf{q}k\,\Theta,\zeta) \simeq \frac{2^{\mu+1}}{\pi\Theta^2}\Gamma^2(1+\mu/2)\,|\mathbf{q}|^{2-\mu}\zeta^2 \; ; \; q < \frac{1}{\zeta} . \qquad (3.26)$$

Thus the spectrum goes to zero quadratically in the parameter ζ, and as the $(2\text{-}\mu)$ power of the wavenumber.

In the high frequency limit the intensity spectrum can be evaluated by segmenting the integral (3.25) into a part $v \le \zeta q$ and a part $v > \zeta q$:

$$k^2 P_I(\mathbf{q},\zeta) = \frac{1}{\Theta^2}\int_0^{2\pi} d\varphi \left\{ \int_0^{\zeta q} v dv \cos(\mathbf{v}\cdot\mathbf{q})\, e^{-f(\mathbf{v},\zeta\mathbf{q})} + \int_{\zeta q}^{\infty} v dv \cos(\mathbf{v}\cdot\mathbf{q})\, e^{-f(\mathbf{v},\zeta\mathbf{q})} - \int_0^{\infty} v dv \cos(\mathbf{v}\cdot\mathbf{q}) \right\} \qquad (3.27)$$

In the first integral we expand $f(\mathbf{v},\zeta\mathbf{q})$ for $\zeta \to \infty$ and obtain

$$\lim_{\zeta\to\infty} f(\mathbf{v},\zeta\mathbf{q}) \sim v^{\mu}$$

in the second integral since $v > \zeta q$, we use

$$\lim_{\zeta\to 0} f(\mathbf{v},\zeta q) \sim 0 .$$

Thus (3.27) yields

$$\lim_{q\to\infty} k^2 P_I(\mathbf{q},\zeta) = \frac{1}{2^{2/\mu}} P_u(\mathbf{q}/2^{1/\mu})$$

$$= \frac{1}{2^{\frac{1}{2-D}}} P_u(\mathbf{q}/ 2^{\frac{1}{4-2D}}) \tag{3.28}$$

so that the intensity spectrum saturates far from the phase screen resulting in a scaled version of the spectrum of u [cf. (3.14)].

As a final comment on the intensity spectrum we observe that (3.25) itself satisfies a scaling relation. Recalling that $\mathbf{q} = \mathbf{K}/k\Theta$ in (3.25) and defining

$$k^2 P_I(\mathbf{K}/k,\Theta,\zeta) \equiv \int \frac{d^2v}{\Theta^2} \cos(\mathbf{v} \cdot \mathbf{K}/k\Theta) \left[e^{-f(\mathbf{v},\zeta\mathbf{K}/k\Theta)} - 1\right] \tag{3.29}$$

we obtain the scaling law

$$P_I(\mathbf{K}/k,\Theta,\zeta) = P_I(\mathbf{K}/k\Theta,1,\zeta\Theta)/\Theta^2 . \tag{3.30}$$

Thus as the phase surface deformation increases, as measured by the parameter Θ, spectral information appears closer to the phase screen and farther from the z-axis, i.e. at larger angles.

4. SCINTILLATION INDEX

A parameter often used to measure the strength of the intensity fluctuations is the scintillation index S_4 defined by the integrated intensity spectrum:

$$S_4^2 = \frac{1}{(2\pi)^2} \int P_I(\mathbf{K},\zeta)\, d^2K \quad . \tag{4.1}$$

Substituting (3.24) into (4.1) we obtain an expression for the scintillation index in terms of the fourth moment of u, i.e.

$$S_4^2 = \langle I_2(\mathbf{R} = 0, z) \rangle - 1 \quad . \tag{4.2}$$

Rumsey[24] introduced an intensity randomization factor U, obtained from (4.1) by replacing $P_I(\mathbf{K},\zeta)$ with

$$\bar{P}_I(\mathbf{K},\zeta) = 4k^2\,\Delta L\, \sin^2(K^2 z/2k)\, \Phi_n(\mathbf{K}) \tag{4.3}$$

where Δ L is the thickness of the phase screen and $\Phi_n(\mathbf{K})$ is the spectrum of irregularities in the screen. Introducing a turbulence strength parameter T by $\Phi_n(\mathbf{K}) = 2\pi T/[k^2 \Delta L\, |\mathbf{K}|^{\mu+2}]$ the intensity randomization factor can be written

$$U = \frac{2^{2-\mu/2}}{(k\Theta)^\mu}\, T \int_0^\infty q^{-\mu-1}\, dq\, \sin^2(\zeta q^2)$$

$$= T\, \frac{2}{\mu} \left[\frac{\zeta^{1/2}}{k\Theta}\right]^\mu \Gamma(1-\mu/2) \cos(\mu\pi/4) \tag{4.4}$$

in agreement with Rumsey since $(\zeta^{1/2}/k\Theta)^\mu = (z/k)^{\mu/2}$. In his analysis of strong scattering from a power-law phase screen Rino[25] determined that $S_4^2 > U$ for all values of μ except in a small region around μ=2. (This corresponds to his parameter value $\nu = 1.5$.) For the fractal dimension in the interval $1 \le D \le 2$, the ν parameter is confined to the interval $0.5 \le \nu \le 1.5$, i.e. ν = 2.5-D. At the end of the ν interval where $U > S_4^2$, the

fractal dimensionality is marginal, i.e. D = 1. The marginal fractal separates diffractals from ordinary random waves. The physical properties of these two types of waves are quite distinct as indicated by the discontinuity in the scintillation index obtained by Rino.[25] This corresponds to $\mu = 2$ in (3.19) and results in $\langle I_2(\mathbf{R},z)\rangle = 1$, so that the saturation level $\langle I_2(\mathbf{r},z)\rangle = 2$ is only obtained if the $\zeta \to \infty$ limit is taken before the $\mu \to 2$ limit [cf. (4.12) below].

In the present formulation the scintillation index (4.2) can be written, using (3.22) with $\mathbf{R} = 0$ and the symmetry properties of the integral, as

$$S_4^2 = \frac{1}{(2\pi)^2} \operatorname{Re} \int d^2u \int d^2v \exp\left\{ i\mathbf{u}\cdot\mathbf{v} - \zeta^{\mu/2} f(\mathbf{u},\mathbf{v}) \right\} - 1 \quad . \tag{4.5}$$

In the region close to the edge of the phase screen ($\zeta \to 0$) we can expand the exponential term in the integrand of (4.5) to obtain,

$$S_4^2 = \frac{1}{(2\pi)^2} \operatorname{Re} \int d^2u \int d^2v \; e^{i\mathbf{u}\cdot\mathbf{v}} \cdot \left\{ 1 - \zeta^{\mu/2} \left[2u^\mu - |\mathbf{u}+\mathbf{v}|^\mu \right] \right\} + O(\zeta^\mu) - 1 \quad . \tag{4.6}$$

The first term inside the curly brackets yields $(2\pi)^2\,\delta(\mathbf{u})$ for the $\mathbf{v}$ - integration. The $\mathbf{u}$-integration of this term yields unity exactly cancelling the -1 factor. The term involving u^μ yields a $\delta(\mathbf{u})$ for the $\mathbf{v}$-integration and $u^\mu\,\delta(\mathbf{u})$ vanishes for the $\mathbf{u}$–integration. The integral involving $|\mathbf{u}+\mathbf{v}|^\mu$ can be transformed using $\mathbf{x} = \frac{1}{2}(\mathbf{u}-\mathbf{v})$ and $\mathbf{y} = \frac{1}{2}(\mathbf{u}+\mathbf{v})$ to obtain

$$S_4^2 = \frac{2^\mu \zeta^{\mu/2}}{(2\pi)^2} \operatorname{Re} \int d^2y \; e^{-iy^2} \int d^2x \, |\mathbf{x}|^\mu e^{ix^2} + O(\zeta^\mu)$$

$$= 2^{\mu+1} \zeta^{\mu/2}\, \Gamma(1+\mu/2) \cos(\mu\pi/4) + O(\zeta^\mu) \quad . \tag{4.7}$$

From the power spectral density (2.12) we observe that Rumsey's turbulence intensity factor can be written[24]

$$T = \mu 2^{\mu-1} k^2 L^{2-\mu} \, \Gamma(\mu/2) \, \Gamma(1+\mu/2) \, \frac{\sin(\mu\pi/2)}{\pi}$$

$$= 2^{\mu-1} \mu (k\Theta)^{\mu} \frac{\Gamma(1+\mu/2)}{\Gamma(1-\mu/2)} \tag{4.8}$$

which when substituted into (4.4) yields one-half of (4.7). Thus we observe that $S_4^2 = 2U$ in the near field region, i.e. the region close to the phase screen.

If instead of the parameter L introduced in (2.14), we had instead selected C_μ in (2.12) in such a way that the spectral strength is given by C_p as defined by (5) in Rino[25], then

$$T = \frac{k^2 C_p}{2\pi} \tag{4.9a}$$

and

$$C_\mu = \frac{C_p}{\pi} \frac{\Gamma(1-\mu/2)}{\mu 2^{\mu} \Gamma(1+\mu/2)} = L^{2-\mu} . \tag{4.9b}$$

Substituting (4.9a) into (4.4) and (4.9b) into (4.7) again yields $S_4^2 = 2U$, in essential agreement with Rino's result away from the marginal fractal region (D = 1).

Finally we can express (4.7) in terms of the fractal dimension as follows:

$$S_4^2 = 2^{5-2D} \zeta^{2-D} \cos \frac{\pi}{2}(2-D) \, \Gamma(3-D) + O(\zeta^{4-2D}) . \tag{4.10}$$

Equation (4.10) agrees with the one-dimensional result of Berry which for the marginal diffractal $D = 1 + \varepsilon$, $\varepsilon \to 0$, yields

$$S_4^2 = 4\pi \varepsilon \zeta + O(\varepsilon^2 \zeta^2) . \tag{4.11}$$

The scintillation index therefore has an infinitesimally slow linear growth in $\zeta\,(=kz)$ for $D = 1 + \varepsilon$.[16]

Far from the phase screen ($\zeta \to \infty$) the behavior of the scintillation index is quite different. Gochelashvily and Shishov[22] find the scintillation index to be given by

$$S_4^2 = 1 + C_4(\mu)\, \zeta^{\mu - 2} + O\left[\zeta^{-2(2-\mu)}\right] \tag{4.12}$$

where

$$C_4(\mu) \equiv 2^{\mu}\, \Gamma(1 + \mu/2)\, \Gamma(4/\mu - 1) \left\{ \frac{1}{\mu\pi} \Gamma(1 + \mu/2) \sin \mu\pi/2 + \frac{2 - \mu}{4\Gamma(2 - \mu/2)} \right\}$$

$$\cdot \left[2\Gamma(\mu/2)\, \Gamma(1 - \mu/2) \right]^{2(\mu - 2)/\mu} . \tag{4.13}$$

Thus the scintillation index for the fractal surface approaches unity from above as $\zeta^{-2(D-1)}$ [cf. (4.12)] so that the intensity scintillations become saturated with propagation distance and/or with the increased strength of "turbulence" [cf. (4.9)].

We now use the scintillation index to establish that a diffractal *cannot* have Gaussian statistics except perhaps in the region far from the phase screen. To establish this we make the contrary assumption, i.e. that a diffractal can be described by a complex Gaussian wave field, and demonstrate that this assumption leads to a value of S_4^2 that is inconsistent with (4.10) and (4.12). Thus we write the diffractral wave amplitude in terms of its real and imaginary parts

$$u(\mathbf{r}, z) = u_r(\mathbf{r}, z) + i u_i(\mathbf{r}, z) \tag{4.14}$$

where u_r and u_i are real homogeneous Gaussian random variables. Then the mean square amplitude is

$$\langle u^2 \rangle = \langle u_r^2 \rangle - \langle u_i^2 \rangle + 2i \langle u_r u_i \rangle \tag{4.15}$$

where the bracket here denotes an average over the Gaussian distribution. But we know that $\langle u^2 \rangle = 0$ because of the fractal nature of the phase [cf. (3.6)], so that the components of the field are statistically independent

$$\langle u_r u_i \rangle = 0 \tag{4.16a}$$

and their mean square levels are equal

$$<u_r^2> \;=\; <u_i^2> \;=\; \tfrac{1}{2} \qquad (4.16b)$$

where the value $\frac{1}{2}$ stems from the normalization (3.9). Now we can write the fourth moment as

$$\begin{aligned} S_4^2 + 1 &= <|u|^4> \\ &= <u_r^{\,4}> + <u_i^4> + 2<u_r^2\, u_i^2> \end{aligned} \qquad (4.17)$$

and using the well known properties of Gaussian functions[14] we obtain for the scintillation index

$$S_4^2 \;=\; 3<u_r^2>^2 + 3<u_i^2>^2 + 2<u_r^2><u_i^2> + 4<u_i u_r>^2 - 1 \;. \qquad (4.18)$$

Inserting the second moment values from (4.16) into (4.18) we obtain

$$S_4^2 \;=\; 1 \qquad (4.19)$$

for a homogeneous isotropic Gaussian wave field.

However the scintillation index for the diffractal given by (4.12) is

$$S_4^2 \;=\; 1 + \frac{A}{z^{2(D-1)}} + O\left[\frac{1}{z^{4(D-1)}}\right] \qquad (4.20)$$

where the definition of the scaling parameter ζ has been used [cf. (3.20)] and A is a z independent positive definite constant. Since $1 \leq D \leq 2$ the scintillation index (4.20) approaches that for a Gaussian random wave as $z \to \infty$. This implies that the statistics of a diffractal are not Gaussian except *perhaps* in the far field region where the fluctuations are saturated.

5. SUMMARY AND CONCLUSION

In an earlier report[26] we have argued how the existence of different scales of fluctuations in the ionosphere can give rise to the observed power-law spectrum of plasma irregularities. This property of the ionosphere suggests that it can be reasonably well modeled as a region with a fractal index of refraction. Subsequently a wave transmitted through such a medium would become a diffractal as proposed by Berry.[16] We have shown herein that a diffractal results from the diffractive free space propagation away from a boundary of a homogeneous, isotropic statistical phase surface having a power-law spectrum. The index μ of the power-law spectrum at the boundary is related to the fractal dimensionality (D) of the diffractal by $D = (4-\mu)/2$. The statistical properties of the free field were determined by calculating the coherence of the diffractal as well as the correlation function and spectrum of the diffractal intensity. These low order moments indicate that the diffractal can be characterized by scaling laws in the spectrum of the wave amplitude and in the intensity spectrum.

Finally the scintillation index S_4^2, often used as a measure of the strength of the intensity fluctuations, is expressed in terms of the distance from the phase screen boundary. For a wave field with Gaussian statistics $S_4^2=1$ independent of z, and we have shown that a diffractal approaches this value from above as an inverse power of the distance from the phase screen in agreement with Rino.[25] This implies that the statistics of a diffractal are not Gaussian except *perhaps* in the far-field region.

REFERENCES

1. See eg. R. Dashen, J. Math. Phys. **20,** 894 (1979).

2. A. Hewish, Proc. Roy. Soc. Lond. **A209,** 81 (1951); ibid. **A214,** 494 (1952).

3. B.J. Rickett, Nature **221,** 158 (1969).

4. A. Hewish, P.F. Scott and D. Wills, Nature **203,** 1214 (1964).

5. H.G. Booker, J.A. Ratcliffe and D.H. Shinn, Phil. Trans. Roy. Soc. Ser. A, **242,** 579 (1950).

6. J.A. Fejer, Proc. Roy. Soc. A **220,** 455 (1953)

7. S.A. Bowhill, J. Res. Natl. Bur. Stand. **65D,** 275 (1961).

8. K.G. Budden, J. Atmos. Phys. **27,** 155 (1965).

9. C.L. Rufenach, Radio Sci. **10,** 155 (1975).

10. R. Uimeki, C.H. Lin and K.C. Yeh, J. Geophys. Res. **82,** 2752 (1977).

11. R.L. Fante, Radio Sci. **15,** 757 (1980)

12. R.K. Crane, J. Geophys. Res. **81,** 2041 (1976); Proc. IEEE **65,** 180 (1977).

13. C.L. Rino, Radio Sci. **14,** 1135 (1979).

14. A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics Vol. 2,* MIT Press, Cambridge, Mass. (1975).

15. B.B. Mandelbrot, *Fractals: Form, Chance and Dimension,* W.H. Freeman, San Francisco (1977).

16. M.V. Berry, J. Phys. A: Math Gen. **12,** 781 (1979).

17. M.V. Berry and Z.V. Lewis, Proc. Roy. Soc. Lond. A **370,** 459 (1980).

18. B. Hughes, M. Shlesinger and E.W. Montroll, Proc. Nat. Acad. Sci. USA, **78,** (1981).

19. E.W. Montroll and B.J. West, in *Fluctuation Phenomena,* eds. E.W. Montroll and J.L. Lebowitz, North Holland Publ. (1979).

20. V. Seshadri and B.J. West, Proc. Natl. Acad. Sci. USA, **79**, 4501 (1982).

21. B.J. West and V. Seshadri, Physica **113A**, 203 (1982).

22. K.S. Gochelashvily and V.I. Shishov, Opt. Quant. Elect. **7**, 524 (1975).

23. E. Jakeman and J.G. McWhirter, J. Phys. A: Math. Gen. **10**, 1599 (1977).

24. V.H. Rumsey, Rad. Sci. **10**, 107 (1975).

25. C.L. Rino, Rad. Sci. **14**, 1147 (1979).

26. B.J. West and E.W. Montroll, A Scaling Model for Phase Screen Fluctuations, LJI-R-82-216 (1982).

27. K.S. Gochelashvily and V.I. Shishov, Opt. Acta. **18**, 767 (1971).

END

FILMED

3-84

DTIC